

# Repeating Decimal Expansions in Different Bases

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## Abstract

*Recall that rational numbers are precisely real numbers with repeating decimal expansions. In this paper, we explore expansions in arbitrary bases and provide an online calculator powered by SAGE to perform calculations with rational numbers, find their expansions, and convert between bases. In addition, we study periods of repeating expansions (i.e., lengths of repeating cycles) as a base is varied and discuss  $p$ -adic number expansions as well.*

## 1 Nomenclature and Preinvestigation

Decimal expansions are used everywhere, and many people are familiar with representations of integers in bases other than base 10 (e.g., binary or hexadecimal numbers). However, representations of non-integer real numbers in bases other than 10 tend to be quite mysterious and novel to most. In this paper, we will with increasing sophistication explore such representations.

### 1.1 Nomenclature

We begin with the unusual strategy of providing opening mathematical inquiries. These will motivate a number of the topics in this paper. To simplify all discussions, we will immediately explain notational styles that will be used regularly throughout this paper.

- Since the term “decimal” connotes base 10, and we will be considering values in a number of bases, we will employ the term “becimal” to indicate any arbitrary base or base other than 10 (see [1]).
- Due to some of the nuances in this investigation, rather than denoting the base of a number merely by a subscript, we will enclose the base in a set of brackets (e.g.,  $xyz_{[b]}$ ). This will later assist in readability. Thus,  $2\bullet 147_{[8]}$  means  $2\bullet 147$  in base 8.
- Rather than the word “digits,” we will use “bigits” as analogous to becimals.

- Recall that in a becimal expansion a vinculum (i.e., line above) indicates repeated text. For example,  $12\bullet\overline{34567} = 12\bullet 34567567567 \dots$  where 567 is repeated forever.

Together, this notation will assist in being both clear and mathematically precise and in abbreviating the often used phrase, “in base  $b$ .”

## 1.2 Preinvestigation

Try out some of these problems.

1. For each of the following values, find the numbers that immediately precede and follow the given value:  $AB_{[12]}$ ;  $500_{[6]}$ ;  $109F_{[16]}$ ;  $19A_{[12]}$ ;  $606_{[7]}$ ; and  $ACEA_{[16]}$ .
2. Convert  $17_{[8]}$  to a number in base 10.<sup>1</sup> Convert  $97_{[10]}$  to a number in base 5.<sup>2</sup>
3. Convert a number in base  $x \neq 10$  to another number in base  $y \neq 10$ . (For instance, convert  $1234_{[7]}$  to base 13.) Can this be done without passing through base 10?
4. Explain how  $\left(\frac{4}{5}\right)_{[6]} = \left(\frac{4}{5}\right)_{[13]} = \left(\frac{4}{5}\right)_{[19]}$ , in fact,  $= \left(\frac{4}{5}\right)_{[b]}$  for any base  $b > 5$ .
5. Given  $\left(\frac{x}{y}\right)_{[10]}$ , with  $x$  and  $y$  relatively prime, explain how to determine if the fraction would produce a terminating or repeating becimal. For  $\left(\frac{1}{y}\right)_{[30]}$ , give a value for  $y$  that will produce a terminating decimal and one that would produce a repeating decimal.
6.  $\left(\frac{2}{3}\right)_{[10]} = 0.\overline{6}_{[10]} = 0\bullet 2_{[3]} = \left(\frac{2}{10}\right)_{[3]}$ . Is it reasonable (or merely a trick) that a repeating becimal in one base can become a terminating becimal in another base?
7. Calculate  $\left(\frac{23_{[4]}}{15_{[9]}}\right)^{-11_{[2]}}$ . Ahh, now don't you wish you had a Becimal Calculator!?!? Well, we later provide one for you.<sup>3</sup>
8. What integer is the 3-adic number<sup>4</sup>  $\dots 22210_{[3]} = \overline{210}_{[3]}$ ? How are  $\overline{210}_{[3]}$  and  $20_{[3]}$  related?
9. Which  $p$ -adic numbers have terminating (to the left) expansions?
10. In a prime base, a number with a terminating expansion (terminating to both the left and right) could be interpreted as a  $p$ -adic or a becimal number (base  $p$ ). Could these be different?

## 2 Introduction

In this section we will define the notion of a becimal expansion and establish some basic notations.

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<sup>1</sup>This is **Becimal Calculator** (<https://billcookmath.com/sage/becimalCalculator.html>) Example #1.

<sup>2</sup>This is **Becimal Calculator** Example #2.

<sup>3</sup>This is **Becimal Calculator** Example #3.

<sup>4</sup>Fields of  $p$ -adic numbers are considered later in this paper.

## 2.1 The Simplest Notions

In base 10,  $\frac{1}{3} = 0.\overline{333333} \dots = 0.\overline{3}$ . However, the same value in base 3 can be written as  $0.1_{[3]}$ . Most interestingly, any infinitely repeating becimal in one base can be written as a terminating becimal in another base.

Before we continue, it is important to at least informally define “repeating” becmals. We recognize a repeating becimal to be a number that has a repeating cycle of bigits to the right of the becimal point (e.g.,  $1.\overline{23}_{[5]}$  and  $186.\overline{372752}_{[9]}$ <sup>5</sup>) A repeating cycle differs significantly from a “repeating pattern.” Examples of the later can include  $0.123457689111012 \dots$  and  $0.10110111011110 \dots$ . Note that a cycle is always of constant length and a pattern need not be. By only considering repeating becimal numbers, we end up constraining our investigation to rational numbers.

There are a number of values or purposes in rewriting a repeating becimal into a terminating becimal. First, it avoids the loss of precision resulting from rounding or truncating becimal values. Second, it allows exact arithmetic in either fractional or becimal forms. Third, the understanding builds a deeper comprehension and appreciation for various bases. In this paper, we will introduce a simple and powerful mathematical app (our **Becimal Calculator**) for converting numbers to other bases and for performing arithmetic operations within and across bases.

## 2.2 Defining Terms

We begin by agreeing that

$$123.456_{[10]} = (1 \times 10^2) + (2 \times 10^1) + (3 \times 10^0) + (4 \times 10^{-1}) + (5 \times 10^{-2}) + (6 \times 10^{-2})$$

where the right-hand-side of the equation is written in our native base 10. Therefore, let  $b$ , the *base*, be a fixed integer greater than 1.<sup>6</sup> Every non-negative real number  $r \in \mathbb{R}$  can be expressed as

$$\begin{aligned} r &= (r_m \cdot b^m) + (r_{m-1} \cdot b^{m-1}) + \dots + (r_1 \cdot b_1) + (r_0 \cdot b^0) + (r_{-1} \cdot b^{-1}) + (r_{-2} \cdot b^{-2}) + \dots \\ &= \sum_{i=-\infty}^m r_i \cdot b^i \end{aligned}$$

where  $r_i \in \{0, 1, \dots, b - 1\}$  is the  $i^{\text{th}}$ -becimal bigit. This is often written in terms of a becimal expansion<sup>7</sup>

$$r = r_m r_{m-1} \dots r_1 r_0 \bullet r_{-1} r_{-2} \dots_{[b]}$$

where the subscript,  $[b]$ , at the end can be suppressed when our base  $b$  is understood. Of course, we can also represent negative numbers by affixing a minus sign to the becimal expansion of the absolute value of  $r$ . For simplicity we will focus on non-negative numbers. Any such becimal expansion can be: terminating<sup>8</sup>, repeating, or non-repeating.

<sup>5</sup>Throughout, we will denote repeating becimal cycles in **red** and the portion of the becimal bigits preceding or delaying the repeating cycle in **blue**. This will aid the reader in visualizing the following discussions.

<sup>6</sup>Negative integer bases are also possible, but most frequently seen in the context of mathematical recreations. We will not consider them here.

<sup>7</sup>The reader should note that the juxtaposition of expressions (e.g.,  $r_m r_{m-1}$ ) denotes subsequent bigits in a becimal expansion and not multiplication.

<sup>8</sup>A terminating becimal, however, can be recognized as an expansion with repeating zeros.

**Investigation 2.1** Navigate to our *Becimal Calculator*<sup>9</sup> and select Example #4. This example converts the hexadecimal (i.e., base 16) number ‘1AB•CD’ to binary (i.e., base 2). Notice that this number has a terminating becimal expansion in both bases. Try changing the output base to 3, checking the box labeled ‘becimals in text?’, and clicking ‘Recompute’. Does the expansion still terminate? Which bases give us a terminating becimal expansion?

**Investigation 2.2** Using our *Becimal Calculator*, you can now experiment freestyle with numbers and operation in various bases. You will notice that our *Becimal Calculator* can handle bases of 100 or 1,000 or even 10,000.<sup>10</sup> Try it out. Get comfortable inputting values, performing operations, and converting numbers to other bases. And, as we know some will, try to break our *Becimal Calculator*. Of course, it is breakable. However, we are confident that you will learn much by challenging the app in this manner.

### 3 Existence and Uniqueness

An immediate question is whether a particular becimal expansion in a particular base even exists, and if so, is it unique?<sup>11</sup> It turns out that an expansion always exists and is either unique or *almost unique*. So, what do we mean by “almost unique?” Notice that in base 10

$$0.99999 \dots = 0.\overline{9} = 1.000 \dots = 1,$$

so we see there is a lack of absolute uniqueness. As another example,  $12.34\overline{6}_{[7]} = 12.35_{[7]}$ . In fact, any base  $b$  terminating becimal expansion can always be represented one other way. This essentially comes down to choosing between repeating the digit  $b - 1$  forever or bumping up the last non-repeated digit by 1.<sup>12</sup> For example,  $0.1\overline{2}_{[3]} = 0.2_{[3]}$ ,  $5.1_{[7]} = 5.0\overline{6}_{[7]}$ , and  $1AC.9E_{[16]} = 1AC.9D\overline{F}_{[16]}$ . To avoid such ambiguity, one usually *simplifies* unnecessarily repeating expansions such as  $0.00\overline{9}_{[10]}$  to the simpler  $0.01_{[10]}$ .

In summary, in any base, a terminating becimal (for  $r > 0$ ) can be written in two distinct forms:

$$r_m r_{m-1} \dots r_0 \cdot r_{-1} \dots r_{k-1} r_k \overline{0}_{[b]} = r_m r_{m-1} \dots r_0 \cdot r_{-1} \dots r_{k-1} [r_k - 1] \overline{[b - 1]}_{[b]}.$$

For example,  $A4 \dots F \cdot 9 \dots B7\overline{0}_{[16]} = A4 \dots F \cdot 9 \dots B6\overline{F}_{[16]}$ .

**Example 3.1** Suppose we are working in base 100. (Yes, base 100.) Then

$$\boxed{55} \boxed{66} \cdot \boxed{77} \boxed{88} \boxed{99} \boxed{99} \boxed{99} \dots = \boxed{55} \boxed{66} \cdot \boxed{77} \boxed{88} \overline{\boxed{99}} = \boxed{55} \boxed{66} \cdot \boxed{77} \boxed{89}.$$

This is Example #5 in our *Becimal Calculator*. While the calculator accepts repeating becimal number inputs, when outputting expansions it always chooses a terminating expansion if it can.

<sup>9</sup>This calculator is located at <https://billcookmath.com/sage/becimalCalculator.html>

<sup>10</sup>Beware, in the *Becimal Calculator*, commas are used to separate digits in numbers in bases exceeding  $b = 36$ . For example, the input “1,000” would be interpreted as a 2-digit number whose first digit was 1 and second digit was 0 (i.e., the same as “10”).

<sup>11</sup>To simplify the reading of this paper, we offload the proofs of existence and uniqueness to an appendix (Section 8). This allows us to focus on some ideas that are more entertaining to a general audience.

<sup>12</sup>The technique, demonstrated later, to convert a repeating becimal to a fraction can verify this notion.

On the other hand, all other becimal expansions are in fact unique. Thus we have actual uniqueness when the expansion is a non-terminating becimal in base  $b$  which does not end with an infinite string of  $(b - 1)$ 's.

Notice that the uniqueness of “a given becimal expansion in a particular base” is different from the uniqueness of “a given value.” Considering  $1.\overline{2}_{[10]} = 1.\overline{032}_{[4]}$ , we quickly recognize that, without altering the value, a number’s representation in one base can appear quite different in another base.

**Investigation 3.2** Use our **Becimal Calculator**<sup>13</sup> to experiment with converting a number in one base to another base. Experiment with large, small, and becimal numbers. Convert from base 10 to base  $x$ , base  $y$  to base 10, and base  $x$  to base  $y$  without passing through base 10. Record your observations, generalizations, and hypotheses.

**Investigation 3.3** Prior to reading the next section (yes, this is a tease), use our **Becimal Calculator** and experiment with numbers with repeating becmals. The app automatically converts repeating becmals in any base to its representative fraction. Record your observations, generalizations, and hypotheses.

- (a) Experiment with base 10 simple periodic<sup>14</sup> becmals. Enter your own numbers into the **Becimal Calculator** and see what results. By experimentation and analysis, for any given repeating becimal expansion, determine an equivalent fraction form without performing the commonly associated calculations.
- (b) Experiment with base  $b$  simple periodic becmals. Enter your own numbers into the **Becimal Calculator** and see what results. By experimentation and analysis, for any given repeating becimal expansion, determine an equivalent fraction form without performing the commonly associated calculations.
- (c) Are you ready to make some generalizations regarding representative fraction forms of simple periodic becmals? If so, great. If not, let us continue to read and learn.

## 4 Fractions and Terminating or Repeating Becimals

Becimal expansions terminate or repeat exactly when we have a rational number. In other words, any quotient of integers,  $x/y$ , where  $y \neq 0$ , will produce a terminating or repeating becimal representation in any base. Conversely, a terminating or repeating becimal can always be rewritten as a fraction of integers.

Notably, the preceding statement has two cases. We develop them as examples in base 10 before generalizing. Let us first consider the terminating expansion  $1.234_{[10]}$ . It is trivial to recognize that  $1.234_{[10]} = \frac{1234}{1000}_{[10]}$ . We have, just that simply, rewritten a terminating becimal as a quotient of integers. Similarly,  $2.3415_{[6]} = \frac{23415}{10^4}_{[6]}$  (where the denominator’s  $10^4_{[6]}$  is  $6^4$ ). Lastly, the expansion  $rs.tuvwxyz_{[b]} = \frac{rstuvwxyz}{10^7}_{[b]}$  (again,  $10^7_{[b]}$  is the same as  $b^7$ ). Thus, a terminating becimal in any base can be written as a quotient of integers.

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<sup>13</sup><https://billcookmath.com/sage/becimalCalculator.html>

<sup>14</sup>Simple periodic becmals begin their repeating cycle immediately to the right of the becimal point. For instance,  $1.\overline{23}$  and  $243.\overline{9523}$  are simple periodic becmals, whereas  $4.\overline{018}$  is a delayed periodic becimal.

So, what about converting a repeating decimal into a fraction? We can perform this conversion in four steps. In particular, to convert some repeating decimal base 10 number  $r$ , we do the following:

**Step 1:** Multiply  $r_{[10]}$  by some power of 10 such that the decimal point moves to the left-hand side of the first repeating cycle.

**Step 2:** Multiply  $r_{[10]}$  by some power of 10 such that the decimal point moves to the right-hand side of the first repeating cycle.

**Step 3:** Subtract Step 1 from Step 2.

**Step 4:** Solve for  $r$ .

Let us look at a specific example.

**Example 4.1** Consider the real number  $r = 1.0\overline{2323} \cdots_{[10]}$ .

$$\begin{array}{rcl}
 r & = & 1.0\overline{23} \\
 \text{Step 1:} & 10r & = 10.0\overline{23} \\
 \text{Step 2:} & 1000r & = 1023.0\overline{23} \\
 & - 10r & = - 10.0\overline{23} \\
 \hline
 \text{Step 3:} & 990r & = 1013 \\
 \text{Step 4:} & r & = \frac{1013}{990}.
 \end{array}$$

Let us now address this process in a more generalized form. Consider a repeating decimal expansion

$$r = a_m \cdots a_0 \cdot d_1 \cdots d_s \overline{c_1 \cdots c_p}_{[b]}$$

in some fixed base  $b$ . Note that we have  $s$  non-repeating digits after the decimal point and a cycle of  $p$  repeating digits. To write  $r_{[b]}$  as a fraction do the following:

**Step 1:** Multiply  $r_{[b]}$  by  $b^s$  so that we *shift* the decimal point by  $s$  digits so that it moves to the left-hand side of the first repeating cycle.

**Step 2:** Multiply  $b^s \cdot r_{[b]}$  by  $b^p$  so that the decimal point moves  $p$  more digits over so it is to the right-hand side of the first repeating cycle.

**Step 3:** Subtract Step 1 from Step 2 to get an integer  $x = (b^p - 1) \cdot b^s \cdot r_{[b]}$

**Step 4:** Solve and get  $r_{[b]} = \frac{x}{(b^p - 1) \cdot b^s}$

To demonstrate this we give a generic example.

**Example 4.2** Let  $r = ef\bullet ghj\overline{klmn}_{[b]}$  be an expansion of a number  $r$  in some fixed base  $b$ .

$$\begin{array}{rcl}
 & r = & ef\bullet ghj\overline{klmn}_{[b]} \\
 \text{Step 1:} & b^3 \cdot r = & efghj\bullet\overline{klmn}_{[b]} \\
 \text{Step 2:} & b^4 \cdot b^3 \cdot r = & efghjklmn\bullet\overline{klmn}_{[b]} \\
 & - \quad b^3 \cdot r = - & efghj\bullet\overline{klmn}_{[b]} \\
 \hline
 \text{Step 3:} & (b^4 - 1) \cdot b^3 \cdot r = & efghjklmn_{[b]} - efghj_{[b]} \\
 \\
 \text{Step 4:} & r = & \frac{efghjklmn_{[b]} - efghj_{[b]}}{(b^4 - 1)b^3}.
 \end{array}$$

This demonstrates that any number in any base with a repeating becimal expansion must be a rational number (i.e., can be written as a ratio of integers). Altogether, then, real numbers with repeating or terminating becimal expansions are precisely the field of rational numbers<sup>15</sup> Moreover, if a number has a repeating or terminating expansion in one base, then it must have repeating or terminating expansions in every other base! (Ah, but wait! We are not saying that a terminating becimal in one base becomes a terminating becimal in another base, nor that a repeating becimal in one base becomes a repeating becimal in another base. This is far from correct. Indeed, the next investigation allows the reader to investigate this further.)

**Investigation 4.3** Previously, we stated that  $\frac{1}{3}_{[10]} = 0.\overline{3}_{[10]} = 0.1_{[3]}$ . Note that a repeating becimal in base 10 is written as a terminating becimal in base 3.

- Experiment with our **Becimal Calculator** and find ten repeating becimals in a base that can be written as terminating becimals in another base.
- Find a repeating becimal in one base that can be written as a terminating becimal in three other bases.
- Write a hypothesis regarding the necessary conditions for when a repeating becimal in one base will be a terminating becimal in another base.

## 5 The Becimal Delay and Cycle Length

We now encourage you to return to our **Becimal Calculator** to consider a new line of experimentation and discovery.

**Investigation 5.1** Let us investigate fractions that produce repeating becimal cycles.

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<sup>15</sup>Let  $T = \{\text{all terminating becimals}\}$  and  $R_p = \{\text{all repeating becimals}\}$ . Then we can say that  $T \cup R_p = \mathbb{Q}$ .

- *One at a time, enter base 10 fractions into the **Becimal Calculator**. Investigate if the resulting becimal is terminating or repeating.* (Hint. Write the fraction in reduced form, and show the numerator and denominator as products of primes.<sup>16</sup> We hope that we are not giving too much away too quickly.)
- *One at a time, enter base  $b$  fractions into the **Becimal Calculator**. Investigate if the resulting becimal is terminating or repeating.*
- *Are you yet able to consider the base of your fraction together with its respective factored denominator and determine if the resulting becimal will terminate or repeat?* (If not, don't worry. We are getting there.)

As a result of your experimentation, you probably noticed that some of your fractions produced terminating becmals and others produced repeating becmals with or without a delay. Indeed, a number of questions may have arisen from your investigation:

- Under what conditions will the fraction produce a terminating becimal?
- Under what conditions will the fraction produce a repeating becimal?
  - Considering the fraction, can the length of the delay prior to the repeating cycle of bigits be predicted?
  - Considering the fraction, can the length of the repeating cycle be predicted?

Prior to addressing these questions (in following sections), we now establish some terminology and notation. As before, let  $b$  be some fixed base and  $r$  some non-negative rational number, say  $r = x/y$  where  $x$  and  $y$  are non-negative, relatively prime integers (i.e.,  $x/y$  is a reduced fraction). We know that  $r$  has a becimal expansion of the form:

$$r = a_m \cdots a_0 \bullet d_1 \cdots d_s \overline{c_1 \cdots c_p}_{[b]}.$$

We refer to the  $a_m \cdots a_0$  portion of the expansion as the *whole part* and  $d_1 \cdots d_s$  the *delay* (before the repeating part). The repeating part forms a *cycle*  $c_1 \cdots c_p$ . If this expansion is written as efficiently as possible, the length of the cycle (i.e.,  $p$ ) is called the *period* of the cycle.<sup>17</sup> As a convention, numbers with terminating expansions will be considered to have period  $p = 0$ . For example, since  $1 = 0.\overline{9}_{[10]}$ , 1 has a period of 0, and  $0.\overline{9}_{[10]}$  has period 1, the most efficient representation has period 0 and no delay.

Notice that for a simple-periodic becimal (with no delay) in base 10:

$$\frac{x}{y} = 0.\overline{c_1 c_2 c_3 \cdots c_p}_{[10]} = \frac{c_1 c_2 c_3 \cdots c_p}_{10^p - 1}.$$

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<sup>16</sup>Some students are initially surprised about primes in different bases. Let  $x_{[b]}$  be prime. If  $x_{[b]}$  is prime and  $x_{[b]} = y_{[c]}$ , then  $y_{[c]}$  is also prime. In other words, being prime does not depend on our becimal representation.

<sup>17</sup> Instead of carefully defining *most efficiently*, we simply note that  $12.\overline{34\overline{564}}_{[10]}$ ,  $12.\overline{34564\overline{56}}_{[10]}$ , and  $12.\overline{34\overline{56}}_{[10]}$  all represent the same number but only the last representation is optimally efficient. Thus this number has a delay of length 1 and a period of 3.



For a base 10 representation with a delay but no whole part we have:

$$\frac{x}{y} = 0.\bullet d_1 d_2 \cdots d_s \overline{c_1 c_2 \cdots c_p}_{[10]} = \frac{d_1 d_2 \cdots d_s c_1 c_2 \cdots c_p [10] - d_1 d_2 d_3 \cdots d_s [10]}{10^s (10^p - 1)}$$

More generally, in base  $b$  we have:

$$\frac{x}{y} = 0.\bullet c_1 c_2 c_3 \cdots c_p [b] = \frac{c_1 c_2 c_3 \cdots c_p [b]}{b^p - 1}$$

for a simple-periodic becimal (with no delay), and

$$\frac{x}{y} = 0.\bullet d_1 d_2 \cdots d_s \overline{c_1 c_2 \cdots c_p}_{[b]} = \frac{d_1 d_2 \cdots d_s c_1 c_2 \cdots c_p [b] - d_1 d_2 d_3 \cdots d_s [b]}{b^s (b^p - 1)}$$

for a delayed-periodic becimal with no whole part.

Wait. Did you see it? These equations are simply generalized forms of the steps and examples provided in **Section 4**. That is correct. All we are doing is multiplying  $x/y$  by powers of  $b$  in order to move the becimal point and align the repeating cycles to subtract away.

## 5.1 Length of the Becimal Delay

We now have most of the tools in place to determine the length of the delay prior to a becimal's repeating cycle. First, we see that  $r$  has a terminating becimal expansion (i.e., period 0) only if  $r = \frac{x}{y}$  and  $y$  divides our base  $b$  to some power. In other words, we can clear out the denominator by multiplying our fraction with a large enough power of our base, say  $b^s$ . Multiplying  $r$  by  $b^s$  shifts our becimal point over  $s$  bigits or places.<sup>18</sup> For instance,  $b^s \cdot (0.\bullet d_1 d_2 \cdots d_s [b]) = d_1 d_2 \cdots d_s \bullet 0 [b]$ . Thus  $s$  is the length of the delayed portion.

For (non-terminating) repeating representations, it turns out that the delay is determined in much the same way. In the appendix (Section 8), we provide a proof of the existence of a repeating expansion for any rational number represented in any base. In doing so, we also show that length of the delayed portion is  $s$  when  $b^s$  is the smallest power of  $b$  such that if  $b^s \cdot r$  is written as a reduced fraction  $x/y$ , then the denominator  $y$  is relatively prime to  $b$ .

**Example 5.2** Working in base  $b = 10$ , consider  $r = \frac{7}{60}$ . For  $b^s = 10^2$ ,  $10^2 \cdot r = \frac{700}{60} = \frac{35}{3}$  where 3 is relatively prime to  $b = 10$ . Since we needed to multiply by  $10^2$  to produce a denominator that is relatively prime to 10, our delay will be  $s = 2$ . In fact,  $r = 0.\bullet 11\overline{6}_{[10]}$ .

Next, considering the same number  $r = \frac{7}{60}$ ,  $5^1 \cdot r = \frac{35}{60} = \frac{7}{12}$ . Now note that 5 is relatively prime to 12. Thus the representation of  $r$  in base  $b = 5$  should have a delay of 1 since it only took  $5^1$  to produce a denominator relatively prime to 5. In fact,  $r = 0.\bullet 0\overline{24}_{[5]}$ .

**Example 5.3** Working in base  $b = 10$ , consider  $r = \frac{4865}{270} = \frac{973}{54} = \frac{973}{2 \cdot 3^3}$ . We need  $b^s = 10^1$  to get  $10^1 \cdot r = \frac{4865}{27}$  so that the denominator is relatively to 10. This means our base 10 expansion will have a delay of length 1. In fact,  $r = 18.0\overline{185}$ .

<sup>18</sup>We used  $s$  in this case to denote the "shift".

In base  $b = 3$ , we will need  $b^s = 3^3$  to make our denominator relatively prime to our base:  $3^3 \cdot \frac{973}{54} = \frac{973}{2}$ . Thus in base  $b = 3$  our expansion will have a delay of  $s = 3$ . In fact,  $r = 200.000\bar{1}_{[3]}$ .

Most generally, we have 3 different delay scenarios for this number  $r$ . If  $b$  is odd and not divisible by 3, there will be no delay. If  $b$  is even and not divisible by 3, there will be a delay of length  $s = 1$ . Finally, if the base is divisible by 3, there will be a delay of  $s = 3$ .

**Investigation 5.4** The length of the delay is entirely dependent on the denominator of the reduced fraction representing our rational number. See if you can create a rational number whose expansions in different bases have delays of lengths 0, 1, 2, and 3 (and no other possible delays). How many prime factors did you need in your denominator? Could you have more factors? Fewer?

Note that you can always pick a base where the decimal has delay length  $s = 0$ . Other than that restriction, can you create rational numbers with arbitrary lists of delay lengths?

## 5.2 Bigit Length of the Cycle

The period, or the length of the repeating cycle, is more difficult. First, suppose we have cleared out the delayed portion and are left with  $\frac{x}{y}$  where  $y$  is now relatively prime to our base  $b$ . If  $y = 1$ , then  $\frac{x}{y} = x$  and our period is 0. So we now assume  $y > 1$ .

The proof in the appendix (Section 8.3) reveals that our period is precisely the smallest power of  $b$  that is congruent to 1 when reduced modulo  $y$  (i.e., the smallest positive exponent  $k$  such that  $b^k - 1$  is divisible by  $y$ ). In the terminology of group theory, the period is equal to the order of  $b$  when  $b$  is considered as an element of the unit group of  $\mathbb{Z}_y$ .<sup>19</sup>

**Example 5.5** Let us return to  $r = \frac{7}{60}_{[10]}$ . As we did in Example 5.2, clearing out the delayed portion of the decimal (by multiplying by  $10^2$ ), we have  $\frac{35}{3}$ . Since  $b = 10$  is congruent to 1 modulo 3,<sup>20</sup> we conclude that  $b$  has order 1 in  $U(y)$ . Therefore, the decimal representation of  $r$  has a repeating cycle or period of length 1 digit (as we have already seen).

On the other hand, working in another base produces a different result. For example, let us look again at  $r = \frac{7}{60}_{[10]}$ . We will multiply by  $5^1$ , producing  $5^1 \cdot r = \frac{35}{60} = \frac{7}{12}$ . (Top Secret: Multiplying  $r$  by 5 will essentially clear out the delayed portion of the respective decimal in base 5. Since  $5_{[10]} = 10_{[5]}$  and  $r = 0.11\bar{6}_{[10]} = 0.02\bar{4}_{[5]}$ , then  $10_{[10]} \cdot 0.11\bar{6}_{[10]} = 1.1\bar{6}_{[10]} = 10_{[5]} \cdot 0.02\bar{4}_{[5]} = 0.2\bar{4}_{[5]}$ .) Notice that 5 is not congruent to 1 modulo 12. However,  $5^2 = 25$  is congruent to 1 modulo 12. Therefore, the base 5 decimal representation of  $r$  has a repeated cycle of length 2 bigits (as we saw above).

**Example 5.6** This time return to  $r = \frac{973}{54}_{[10]}$ . As we did in Example 5.3, clearing out the delayed portion of the decimal (by multiplying by  $10^1$ ), we have  $\frac{4865}{27}$ . Checking what 10,  $10^2$ , and  $10^3$  are modulo 27, we see that  $b^p = 10^3$  is the smallest power yielding a result congruent to 1 modulo 27. This means that the decimal expansion of  $r$  has period 3. In fact, recall that  $r = 18.0\bar{185}$ .

On the other hand, when considering base  $b = 3$ , it took  $b^s = 3^3$  to clear out the delay portion of  $r$ . This left us with  $\frac{973}{2}$ . Notice that  $b^p = 3^1 = 3$  is congruent to 1 modulo 2, so  $r$ 's base 3 decimal expansion should have a period of 1. Again recall,  $r = 200.000\bar{1}_{[3]}$ .

<sup>19</sup>The unit group of  $\mathbb{Z}_y$  is commonly denoted by  $U(y)$ . It consists of equivalence classes (modulo  $y$ ) represented by integers relatively prime to  $y$ . The group operation is multiplication (modulo  $y$ ).

<sup>20</sup>We often write this  $10 \equiv 1 \pmod{3}$

If we chose base  $b = 54$ , then  $r$  would have a terminating expansion:  $18.\overline{1}_{[54]}$  and thus a period of  $p = 0$ .

[ In summary, if  $r = \frac{x}{y}_{[b]}$  is a reduced fraction, the portion of the denominator  $y$  that shares factors with  $b$  determines the length of the becimal delay and the portion of the denominator that is relatively prime to  $b$  determines the period (i.e., the length of the repeating becimal cycle). ]

**Investigation 5.7** Use the **Becimal Calculator** to compute becimal expansions of various fractions to see what kinds of periods are possible. Can you always guarantee a period 0 or period 1 representation?

**Investigation 5.8** Assume that you have a repeating becimal in base  $b$  with a delay of  $s$  bigits and a period of  $p$ . Create a fraction for which the becimal expansion would have:

- base  $b = 10$  with a delay of  $s = 3$  bigits and a period of  $p = 5$ ;
- base  $b = 7$  with a delay of  $s = 2$  bigits and a period of  $p = 4$ ; and
- base  $b = 3$  with a delay of  $s = 1$  bigits and a period of  $p = 2$ .
- Are there any combinations of values for  $b$ ,  $s$ , and  $p$  such that are impossible?

We find that the possible periods of becimal expansions of a particular fraction are linked to the structure of an associated unit group. Before exploring this more fully, we refer the interested reader to Appendix 8.4 for more details about such unit groups:  $U(y)$ .

Suppose  $r = \frac{x}{y}$  is a reduced fraction and  $y$  is prime. Then since  $U(y)$  is a cyclic group of order  $y - 1$ , it turns out that  $r$ 's becimal representations must either terminate (i.e., have period 0) or have a period dividing  $y - 1$ , and all such divisors are achievable.

**Example 5.9** Consider  $r = \frac{2}{7}$ . Notice that  $y = 7$  yields a unit group with elements of orders 1, 2, 3, and 6. Thus by choosing our base carefully, we can find becimal representations of  $r$  with period 0, 1, 2, 3, and 6 (and no other). You can experiment with the **Becimal Calculator** to find such expansions. Here are some possible answers:

- Period 0:  $r = 0.\overline{2}_{[7]}$
- Period 1:  $b = 8$  has order 1 modulo 7, so  $r = 0.\overline{2}_{[8]}$  is a period 1 expansion.
- Period 2:  $b = 6$  has order 2 modulo 7 (since  $6^2 = 36_{[10]}$  is congruent to 1 mod 7 but  $6^1 = 6$  is not congruent to 1 mod 7), so  $r = 0.\overline{14}_{[6]}$  is a period 2 expansion.
- Period 3:  $b = 2$  has order 3 modulo 7 (since  $2^3 = 8$  is congruent to 1 mod 7 but  $2^1 = 2$  and  $2^2 = 4$  are not), so  $r = 0.\overline{010}_{[2]}$  is a period 3 expansion.
- Period 6:  $b = 10$  has order 6 modulo 7, so  $r = 0.\overline{285714}_{[10]}$  is a period 6 expansion.

Students who understand the structure of the unit groups<sup>21</sup> of integers mod  $y$  should be able to determine the possible element orders in  $U(y)$  and be able to find bases yielding representations of  $x/y$  with various period lengths.

**Example 5.10** Again, as in Examples 5.2 and 5.5, consider the number  $r = \frac{7}{60}_{[10]}$ . Let us see what can happen in various bases.

In some bases, we might share some factors with the denominator  $60_{[10]}$ . So the part of the denominator relatively prime to the base (i.e., the part responsible for the period) must end up being a divisor of  $60_{[10]}$ .

The groups  $U(y)$  where  $y$  is a divisor of  $60_{[10]}$  only have elements of orders 1, 2, and 4. This implies that representations of  $r = \frac{7}{60}_{[10]}$  can have periods  $p = 0, 1, 2,$  and 4 but nothing else.<sup>22</sup>

However, even more can be said. Since  $60_{[10]} = 2^2 \cdot 3 \cdot 5$ , no matter our choice of base  $b$ , either  $r$  or  $b \cdot r$  or at worst  $b^2 \cdot r$  will reduce to a fraction whose denominator is relatively prime to  $b$ . This follows from the fact that the largest power appearing in the prime factorization of  $r$ 's denominator (i.e.  $60_{[10]} = 2^2 \cdot 3 \cdot 5$ ) is 2. Consequently, no matter how we choose our base  $b$ , the largest delay we can see is  $s = 2$ .

In particular, in base  $b = 6$ , we need to multiply  $r$  by  $b^2$  to have a denominator relatively prime to  $b = 6$ . This leaves us with  $b^2 \cdot r = \frac{21_{[10]}}{5}$ ; so  $y = 5$ . Notice that  $b = 6$  is congruent to 1 modulo  $y = 5$ . Thus because  $b = 6$  has order 1 in  $U(5)$ , our representation has a period of 1. In fact,  $r = 0.04\bar{1}_{[6]}$ .

**Example 5.11** Once again, as in Examples 5.3 and 5.6, consider  $r = \frac{973}{54}_{[10]}$ . Analyzing the structure of  $U(54)$  will allow us to determine the possible period lengths appearing in becimal expansions of  $r$ .

Since  $54 = 2 \cdot 3^3$ , the group structure of  $U(54)$  is the same as  $U(2) \times U(27)$ . But  $U(2)$  is trivial and  $U(27)$  is a cyclic group of order  $3^3 - 3^2 = 27 - 9 = 18$ . Therefore,  $U(54)$  has elements of orders 1, 2, 3, 6, 9, and 18 (i.e., divisors of 18). Therefore, by choosing a base which manifests as an element of one of these orders (working modulo 54), we can find becimal expansions of  $r$  with these periods. No other periods are possible except for  $p = 0$ . (We can always make the expansion terminate.)

**Investigation 5.12** Use our **Becimal Calculator** and verify that we get periods  $p = 0, 1, 2, 3, 6, 9,$  and 18 when expanding  $r = \frac{973}{54}_{[10]}$  in bases  $b = 6, 3, 26, 10, 17, 7,$  and 5 respectively.

## 6 Early Conclusion

Before we consider the bonus topic of  $p$ -adic numbers, let us very briefly recap some previous topics of investigation. Some of these include:

- the existence of terminating or repeating forms of rational numbers;
- the *almost* uniqueness of terminating becmals and the uniqueness of repeating becmals;
- the converting of a terminating or repeating becimal in one base to the other form in another base;

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<sup>21</sup>More information regarding unit groups is provided in the appendix (Section 8.4).

<sup>22</sup>These periods can be achieved using bases  $b = 60, 10, 4,$  and 3 respectively.

- rewriting a terminating or repeating decimal into its associated fractional representation; and
- the delay and cycle length (i.e., period) of repeating decimals.

Many high school and college mathematics students would find much of this material quite interesting. However, to further deepen understanding of these interconnected ideas, we post one more investigation.

**Investigation 6.1** *Note that our **Becimal Calculator** provides a calculation and readout that we have not discussed. If you click on the checkbox labeled “suggested base,” a suggested base calculation appears after the input and output.*

- Enter some rational numbers in fractional and decimal form and convert them to other bases. Investigate the “suggested base” calculation.

*Prior to this exercise, we have asked for you to only enter one expression at a time into our wonderful **Becimal Calculator**. However, this calculator is a beast, and one expression at a time does not fully display its power and coolness. So...*

- Be bold. Enter a few expressions in different bases (yes, the bases of a number of expressions can be different) and use the operations addition (+), subtraction (−), multiplication (\*), division (/), and exponentiation (^). For example, recall *Preinvestigation (Section 1.2) Exercise #7* to calculate the rather complicated number  $\left(\frac{23_{[4]}}{15_{[9]}}\right)^{-11_{[2]}}$ . This is our **Becimal Calculator**’s built-in Example #3. (Go ahead. Try to break our **Becimal Calculator**. We know you want to try.) Examine the “suggested base” in the calculation string. Do a few examples. What do you notice about the “suggested base” output? (This time we are not providing any clues. Bonus points if you figure it out.)

## 7 Over-time Investigation: P-Adic Numbers

We conclude this paper with a quick trip to the world of  $p$ -adic numbers. In many ways, the existence of decimal expansions gets right at the heart of what the real numbers  $\mathbb{R}$  really are. It turns out that the real numbers can be viewed as a certain kind of completion of the rational numbers  $\mathbb{Q}$ . But what is meant by *completion*? Very roughly, a completion is the result of taking a space and filling in its holes.

In more detail, a *metric space* is a space equipped with a way of measuring distances (called a *metric*). In such spaces, one can speak of convergent sequences and Cauchy sequences. Briefly, convergent sequences are lists of points that get closer and closer to a specific limit point as we go further and further out in that list. Cauchy sequences are also lists of points. But these points merely get closer and closer to each other as we go further and further out in that list. A space is said to be complete (in the metric sense) if every Cauchy sequence converges to a point in the space.

Now we focus on the real numbers. They form a metric space when given the metric distance  $d(x, y) = |x - y|$  (the distance between  $x$  and  $y$  is the magnitude of their difference). Working in  $\mathbb{R}$  we have that a sequence  $a_1, a_2, \dots$  converges to  $L$  if for every positive distance  $\epsilon > 0$  there is some  $N > 0$

such that  $|a_n - L| < \epsilon$  for all  $n \geq N$ . A sequence  $a_1, a_2, \dots$  is *Cauchy* if for every  $\epsilon > 0$  there exists some  $N > 0$  such that  $|a_m - a_n| < \epsilon$  for all  $m, n \geq N$ .

You can see that Cauchy means getting close together while convergence means getting close to a particular point. One can easily show that all convergent sequences (i.e., getting close to some limit) must be Cauchy (i.e., getting close to each other). A complete space is one where the converse is true (i.e., every Cauchy sequence converges).

While the real numbers are a complete space, the rational numbers are not. For example, the rational sequence constructed from digits of  $\pi$ , namely 3, 3.1, 3.14, 3.141,  $\dots$ , is a Cauchy sequence, but it does not converge. Well, at least it does not converge to a rational number!

It turns out that every metric space (a space where we can measure distances) has a minimal completion that is unique up to a natural equivalence (i.e., isometry – that is – a distance preserving invertible function).

This is where things get strange. The absolute value is not the only way to measure distances in the field of rational numbers. Given a prime number  $p$ , one can define the  $p$ -adic norm (a norm is like a generalized absolute value) on  $\mathbb{Q}$  as follows:  $|0|_p = 0$  and if  $r$  is a non-zero rational number, we can write  $r = p^k \cdot \frac{x}{y}$  where  $x$  and  $y$  are not divisible by  $p$  (i.e., factor all powers of  $p$  out of the numerator and denominator and lump them together into  $p^k$ ) where  $k$  is an integer. Here we define  $|r|_p = p^{-k}$ . For example,  $|36|_3 = |3^2 \times 4|_3 = 3^{-2} = \frac{1}{9}$  and  $|\frac{8}{75}|_5 = |5^{-2} \times \frac{8}{3}|_5 = 5^2 = 25$ . In other words, the  $p$ -adic norm measures how divisible by  $p$  a number is. Numbers divisible by large powers of  $p$  are small. Yes, this is weird.

One can see that  $|r|_p \geq 0$  and  $|r|_p = 0$  if and only if  $r = 0$ . Also,  $|r + s|_p \leq \max\{|r|_p, |s|_p\} \leq |r|_p + |s|_p$ . The latter property is a kind of strong triangle inequality. We call  $|r - s|_p$  the  $p$ -adic distance between  $r$  and  $s$ . It is easy to show that this satisfies the axioms of a metric and thus makes the rational numbers into a metric space. The completion of the rational numbers relative to this metric is called the field of  $p$ -adic numbers and is denoted  $\mathbb{Q}_p$ . These numbers can be added, subtracted, multiplied, and divided (except by 0) just like real numbers.

Interestingly  $p$ -adic numbers can be represented uniquely (no more *almost* uniqueness) as a sum  $r_{-m} \times p^{-m} + \dots + r_{-1} \times p^{-1} + r_0 + r_1 \times p + r_2 \times p^2 \dots$  where  $r_j \in \{0, \dots, p - 1\}$ . It seems wrong to have a sum involving larger and larger powers of  $p$ , but keep in mind that in the  $p$ -adic norm, large powers of  $p$  are small! Such an expansion is denoted  $\dots r_2 r_1 r_0 . r_{-1} \dots r_{-m} [p\text{-adic}]$ .

In a funny twist of fate, a  $p$ -adic number is rational if and only if it has a repeating or terminating  $p$ -adic expansion! This could be denoted by  $r = \overline{c_\ell \dots c_1} d_s \dots d_0 \bullet a_1 \dots a_m [p\text{-adic}]$ .

Real and  $p$ -adic numbers are not compatible in general. However, they do have the rational numbers in common. In both systems, having a repeating expansion is the same as being a rational number. As another happy turn, if a *non-negative* rational number has a terminating expansion in base  $p$  (i.e.,  $r = r_m \dots r_0 \bullet r_{-1} \dots r_{-s} [p]$ ), then this is both a base  $p$  and a  $p$ -adic expansion (i.e., the expansions match):

$$r_m \dots r_0 \bullet r_{-1} \dots r_{-s} [p] = r_m \cdot p^m + r_{m-1} \cdot p^{m-1} + \dots + r_{-s} \cdot p^{-s} = r_m \dots r_0 \bullet r_{-1} \dots r_{-s} [p\text{-adic}].$$

Another odd feature of  $p$ -adic expansions is the lack of minus signs to denote negative numbers. Consider the number  $\dots 222 [3\text{-adic}] = \overline{2} [3\text{-adic}]$ . When adding or multiplying  $p$ -adic representations, one does it digit by digit while carrying  $p$ 's. Thus in the 3-adic numbers we have that  $1 + \overline{2} [3\text{-adic}]$  is

$$1 + (2 + 2 \cdot 3 + 2 \cdot 3^2 + \dots) = 3 + 2 \cdot 3 + 2 \cdot 3^2 = 0 + (1 + 2) \cdot 3 + 2 \cdot 3^2 = 0 + 0 \cdot 3 + (1 + 2) \cdot 3^2 + \dots$$

etc. Therefore,  $1 + \bar{2}_{[3\text{-adic}]} = 0$ . In other words,  $-1 = \cdots 222_{[3\text{-adic}]} = \bar{2}_{[3\text{-adic}]}$  in the 3-adic numbers!

It turns out that while the  $p$ -adic numbers (like the real numbers) are complete, unlike the real numbers, the field of all  $p$ -adic numbers cannot be ordered. In fact,  $p$ -adic analysis, as opposed to real analysis, in many ways more resembles complex analysis. We recommend the very accessible and clearly written introduction to the  $p$ -adic numbers by Katok [2].

**Investigation 7.1** Use the *Becimal Calculator* to explore  $p$ -adic expansions. What is the expansion of  $p^k$ ? How about  $-p^k$ ? What is the relationship between the  $p$ -adic expansion of a positive integer and its negation? The calculator's Example #6 gives an example of entering a  $p$ -adic number with a repeating expansion.

## 8 Appendix: Technicalities and Proofs

In this appendix we provide proofs of the existence and uniqueness of becimal representations. We also show that the rational numbers are precisely those with repeating (or terminating) becimal expansions. Finally, we include some technical background about unit groups:  $U(y)$ .

To begin, let  $b$  be a fixed positive integer greater than 1 and let  $r$  be a fixed real number. If  $r = 0$ , we already know a becimal expansion (i.e., let every bigit be 0). Note that this is its unique representation since if any bigit were nonzero,  $r$  would be nonzero as well. Next, if  $r < 0$ , then we represent  $|r|$  in base  $b$  and place a minus sign at the beginning. Note that this means positive and negative numbers are represented differently. Thus to address existence and uniqueness we can just focus on  $|r|$ . So, going forward, we suppose  $r > 0$ .

### 8.1 Existence

First, we address existence. Since  $b > 1$ ,  $b^2 = b \cdot b > b \cdot 1 = b$  and also  $1 > 1/b = b^{-1}$ . In general,  $\cdots < b^{-2} < b^{-1} < 1 < b < b^2 < b^3 < \cdots$  where, as  $n \rightarrow \infty$ ,  $b^n \rightarrow \infty$  and  $b^{-n} \rightarrow 0$ . Now because  $r > 0$  and  $b^n \rightarrow \infty$  as  $n \rightarrow \infty$  we must have  $r < b^k$  for some  $k$  large enough. Likewise, because  $b^{-n} \rightarrow 0$  as  $n \rightarrow \infty$ , we must have  $b^{-k} < r$  for  $k$  large enough. Consider the set  $\{\ell \in \mathbb{Z} \mid r \geq b^\ell\}$ . Our last two sentences tell us that this set of integers is both bounded above and non-empty. Therefore, it must have a maximum, call it  $m$ .<sup>23</sup>

We now know that  $b^m \leq r$  since  $m$  belongs to our set, but also  $r < b^{m+1}$  because  $m$  was the maximal element in that set. In particular, we have  $0 \leq r < b^{m+1}$  and so  $0 \leq rb^{-m} < b$ . Let  $r_m$  be the maximum integer such that  $0 \leq r_m < b$  and  $r_m \leq rb^{-m}$ . In other words,  $r_m \in \{0, \dots, b-1\}$  and  $0 \leq rb^{-m} - r_m < 1$ . Therefore,  $0 \leq r - r_m b^m < b^m$ .

Likewise, we can select  $r_{m-1} \in \{0, \dots, b-1\}$  so that  $0 \leq r - r_m b^m - r_{m-1} b^{m-1} < b^{m-1}$ . Continuing in this fashion (i.e., by mathematical induction marching down powers of  $b$ ), we have  $0 \leq r - (r_m b^m + r_{m-1} b^{m-1} + \cdots + r_{-N} b^{-N}) < b^{-N}$  for any positive integer  $N$  where each bigit  $r_i$  is selected from  $\{0, \dots, b-1\}$ . Notice that  $b^{-N} \rightarrow 0$  as  $N \rightarrow \infty$ , so (by the squeeze theorem for limits)  $r - \sum_{i=-\infty}^m r_i b^i = 0$ . In other words,  $r = \sum_{i=-\infty}^m r_i b^i = r_m r_{m-1} \cdots r_0 \bullet r_{-1} r_{-2} r_{-3} \cdots$  is our desired becimal expansion.

<sup>23</sup>This is more-or-less the well ordering principal.

## 8.2 Uniquenes

Suppose  $r$  has two becimal representations, say

$$\begin{aligned} r &= r_m r_{m-1} \cdots r_0 \bullet r_{-1} r_{-2} \cdots \\ &= s_\ell s_{\ell-1} \cdots s_0 \bullet s_{-1} s_{-2} \cdots \end{aligned}$$

are representations. We have  $\sum_{i=-\infty}^m r_i \cdot b^i = \sum_{j=-\infty}^{\ell} s_j \cdot b^j$  where  $r_i, s_j \in \{0, \dots, b-1\}$  for all  $i$  and  $j$ . Padding either one out with zeros if necessary, we may assume  $m = \ell$ . Suppose  $k$  is the first bigit where these representations differ (i.e.,  $r_m = s_m, r_{m-1} = s_{m-1}, \dots, r_{k-1} = s_{k-1}$ ) and without loss of generality assume  $r_k > s_k$ .

First, note that we must have  $r_j = 0$  for all  $j > k$  (i.e.,  $r = r_m \cdots r_0 \bullet r_{-1} \cdots r_k \bar{0}$ ). Why? Otherwise,

$$\begin{aligned} r &= r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} r_k r_{k+1} \cdots \\ &> r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} r_k \bar{0} &= s_m \cdots s_0 \bullet s_{-1} \cdots s_{k-1} r_k \bar{0} \\ &&\geq s_m \cdots s_0 \bullet s_{-1} \cdots s_{k-1} s_k s_{k+1} \cdots = r \end{aligned}$$

so that  $r > r$  (which is impossible).

So far we have  $r = r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} r_k \bar{0} = r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} s_k s_{k+1} \cdots$ . Next, we claim that  $s_k = r_k - 1$ . Otherwise, if  $r_k - 1 > s_k$ , we again have a problem:<sup>24</sup>

$$\begin{aligned} r &= r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} r_k \\ &> r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} [r_k - 1] \\ &\geq r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} s_k s_{k+1} \cdots = r. \end{aligned}$$

Thus  $r > r$  (which is impossible).

Finally, we claim that not only  $s_k = r_k - 1$  but also  $s_j = b - 1$  for  $j > k$ . Suppose not. Then for some  $j > k$  we have  $s_j < b - 1$ . Therefore,

$$\begin{aligned} r &= r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} s_k s_{k+1} \cdots s_{j-1} s_j \cdots \\ &= r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} [r_k - 1] [b - 1] \cdots [b - 1] s_j \cdots \\ &< r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} [r_k - 1] [b - 1] \cdots [b - 1] [b - 1] \\ &\leq r_m \cdots r_0 \bullet r_{-1} \cdots r_{k-1} r_k = r \end{aligned}$$

Once again, we have  $r < r$  (which is impossible).

Therefore, either two becimal representations are identical (ignoring possible unnecessary leading zeros) or one of them terminates at the  $j^{\text{th}}$  bigit and the other's  $j^{\text{th}}$  bigit is one less followed by infinitely repeating  $(b - 1)$ 's.

<sup>24</sup>We have placed square braces around the  $k^{\text{th}}$  digit here.



### 8.3 Repeating Becimal and the Rational Numbers

In this subsection we will show that only rational numbers have repeating (or terminating) becimal expansions and also that rational numbers always have repeating (or terminating) becimal expansions.

First, suppose that  $r$  has a repeating becimal expansion:  $r = a_m \cdots a_0 \bullet d_1 \cdots d_s \overline{c_1 \cdots c_p}_{[b]}$ . Therefore,

$$b^s \cdot r = a_m \cdots a_0 d_1 \cdots d_s \bullet \overline{c_1 \cdots c_p}_{[b]} = a_m \cdots a_0 d_1 \cdots d_s \bullet c_1 \cdots c_p \overline{c_1 \cdots c_p}_{[b]}$$

Thus  $b^p \cdot b^s \cdot r = a_m \cdots a_0 d_1 \cdots d_s c_1 \cdots c_p \bullet \overline{c_1 \cdots c_p}_{[b]}$  so that

$$\begin{aligned} b^p \cdot b^s \cdot r - b^s \cdot r &= \left( a_m \cdots a_0 d_1 \cdots d_s c_1 \cdots c_p \bullet \overline{c_1 \cdots c_p}_{[b]} \right) - \left( a_m \cdots a_0 d_1 \cdots d_s \bullet \overline{c_1 \cdots c_p}_{[b]} \right) \\ (b^p - 1) \cdot b^s \cdot r &= \left( a_m \cdots a_0 d_1 \cdots d_s c_1 \cdots c_p \right) - \left( a_m \cdots a_0 d_1 \cdots d_s \right) \end{aligned}$$

Therefore,

$$r = \frac{\left( a_m \cdots a_0 d_1 \cdots d_s c_1 \cdots c_p \right) - \left( a_m \cdots a_0 d_1 \cdots d_s \right)}{(b^p - 1) \cdot b^s}$$

is a rational number.

The converse is much more subtle. We now suppose that  $r = \frac{x}{y}$  is a positive rational number where  $x$  and  $y$  are relatively prime positive integers (i.e.,  $x/y$  is a reduced fraction). We wish to show  $r$  has a terminating or repeating becimal expansion.

First, we note that by multiplying a number  $r$  by a large enough power of our base  $b$  we can cancel off any prime factors shared by  $y$  and  $b$ . In particular, there is some minimal integer  $s \geq 0$  such that  $b^s \cdot r = \frac{x'}{y'}$  where  $x'$  and  $y'$  are relatively prime positive integers and  $y'$  is relatively prime to  $b$ . Also, note that multiplying a number such as  $r$  by  $b^s$  merely shifts the bigits in its expansion  $s$  positions over to the left. This shift of  $s$  bigits will account for the delay portion of the becimal expansion.

Since this shift has no impact on whether we have a repeating expansion or not, we may now assume that  $y$  and  $b$  are relatively prime. If  $y = 1$  (i.e.,  $r = x$  is an integer), we have a terminating expansion and thus are done. Thus suppose  $y > 1$ . We show that in this case  $r$  has a repeating becimal expansion without delay.

Divide (i.e., division with remainder of integers)  $x$  by  $y$  and get  $x = qy + z$  where  $0 \leq z < y$ . Thus  $r = q + \frac{z}{y}$ . Note that  $z \neq 0$  since  $r$  is not an integer. Also, we observe that, from basic number theory,  $\gcd(y, z) = \gcd(x, y) = 1$  so that  $y$  and  $z$  are relatively prime.

We now appeal to basic number theory (or modern algebra). Since  $b$  and  $y$  are relatively prime,  $b^p$  is congruent to 1 modulo  $y$  for large enough  $p$ . Why? Well,  $b$  is in the unit group  $U(y)$  whose order is  $\varphi(y)$  (i.e., Euler's totient function applied to  $y$ ) and so by Lagrange's theorem  $b^{\varphi(y)} = 1$  since the order of  $b$  in  $U(y)$  must divide the order of  $U(y)$ . Suppose  $p$  is the smallest positive integer such that this is true (i.e.,  $p$  is the order of  $b$  in  $U(y)$ ).

Therefore, since  $b^p$  is congruent to 1 modulo  $y$ , there is a (positive) integer  $n$  such that  $b^p - 1 = ny$ . Thus  $(b^p - 1) \frac{z}{y} = (b^p - 1) \frac{nz}{ny} = nz$ . In other words,  $b^p \frac{z}{y} = \frac{z}{y} + nz$ .

We can translate this to a becimal expansion to see that we indeed have a repeating expansion. Suppose  $\frac{z}{y} = 0 \bullet r_1 r_2 \cdots r_{[b]}$  (we have such an expansion since  $0 < z/y < 1$ ) and  $nz = s_\ell \cdots s_{0[b]}$  (we

have such an expansion since  $nz$  is a positive integer). Therefore, in terms of becimal expansions, we have that

$$\begin{aligned} r_1 r_2 \cdots r_p \bullet r_{p+1} r_{p+2} \cdots [b] &= b^p \cdot \frac{z}{y} = nz + \frac{z}{y} \\ &= s_\ell \cdots s_{0[b]} + 0 \bullet r_1 r_2 \cdots [b] = s_\ell \cdots s_0 \bullet r_1 r_2 \cdots [b] \end{aligned}$$

Therefore, not only can we conclude that the becimal expansion of  $nz$  is the integer part of the left hand side (i.e., the becimal  $r_1 \cdots r_p$ ), but also comparing the next bigits we get that  $r_{p+1}$  must match  $r_1$  and  $r_{p+2}$  must match  $r_2$  etc. Therefore,  $r_j = r_{j+p}$  for  $j > 0$ . In other words,  $z/y$  has a repeating becimal expansion. In fact,  $nz$  gives us this repeating part. Thus  $r = q + \frac{z}{y}$  has a repeating becimal expansion.

We can now let  $\frac{z}{y} = 0 \bullet \overline{c_1 \cdots c_p [b]}$  where our cycle,  $c_1 \cdots c_p$ , is written as efficiently as possible (i.e., not repeated). Our final task is to show that the period of this expansion (i.e.,  $p$ ) is indeed the order of  $b$  in the unit group  $U(y)$ . We have that  $b^p$  is the smallest power of  $b$  such that  $(b^p - 1) \cdot \frac{z}{y} = b^p \cdot \frac{z}{y} - \frac{z}{y} = c_1 \cdots c_p \bullet \overline{c_1 \cdots c_p [b]} - 0 \bullet \overline{c_1 \cdots c_p [b]} = c_1 \cdots c_p [b]$  is an integer.

Therefore,  $p$  is the smallest positive integer such that  $(b^p - 1) \cdot z$  is divisible by  $y$ . Recall that  $y$  and  $z$  are relatively prime. Thus, in fact,  $p$  is the smallest positive integer such that  $b^p - 1$  is divisible by  $y$ . This establishes that  $p$  is the order of  $b$  in  $U(y)$ .

In summary, assuming that  $r = \frac{x}{y} = a_m \cdots a_1 \bullet d_1 \cdots d_s \overline{c_1 \cdots c_p [b]}$  is written as efficiently as possible, the length of the delay portion (i.e.,  $s$ ) is the smallest power  $b^s$  such that  $b^s \cdot r = \frac{x'}{y'}$  (written as a reduced fraction) with  $y'$  and  $b$  relatively prime. Moreover, the period of the repeating part (i.e.,  $p$ ) is the smallest power  $b^p$  such that  $b^p$  is congruent to 1 modulo  $y$  (i.e., the order of  $b$  in  $U(y)$ ).

## 8.4 More About Unit Groups

Recall that  $\mathbb{Z}_y$  refers to the ring of integers modulo  $y$  (for some fixed positive integer  $y$ ). The unit group of  $\mathbb{Z}_y$ , denoted  $U(y)$ , consists of equivalence classes modulo  $y$  represented by integers relatively prime to  $y$ . This is a group under multiplication modulo  $y$ .

For example,  $U(12) = \{1, 5, 7, 11\}$  where 5 is actually the equivalence class  $5 + 12\mathbb{Z} = \{\dots, -19, -7, 5, 17, \dots\}$ . Working in  $U(12)$ , we have  $5 \cdot 7 = 11 \pmod{12}$ .

The number of elements in  $U(y)$  is denoted by  $\varphi(y)$ . This function,  $\varphi(y)$ , is called Euler's  $\varphi$ -fuction or the totient function. Thus  $\varphi(y)$  counts the number of integers between 1 and  $y$  that are relatively prime to  $y$ .

The Chinese Remaindering Theorem establishes that  $U(ab)$  and  $U(a) \times U(b)$  are isomorphic as groups whenever  $a$  and  $b$  are relatively prime. Thus,  $U(y)$  has the same group structure as  $U(p_1^{k_1}) \times U(p_2^{k_2}) \times \cdots \times U(p_\ell^{k_\ell})$  if  $p_1^{k_1} p_2^{k_2} \cdots p_\ell^{k_\ell}$  is the prime factorization of  $y$ . Consequently,  $\varphi(y) = \varphi(p_1^{k_1}) \varphi(p_2^{k_2}) \cdots \varphi(p_\ell^{k_\ell})$ . One can show that  $|U(p^k)| = \varphi(p^k) = p^k - p^{k-1}$  for any prime power  $p^k$ . This means we can easily compute  $\varphi(y)$  given we have already factored  $y$  into primes. As we saw above,  $|U(12)| = \varphi(12) = \varphi(2^2 \cdot 3) = \varphi(2^2) \varphi(3) = (2^2 - 2)(3 - 1) = 4$ .

Chinese Remaindering also allows us to reduce the task of understanding the group structure of these unit groups to understanding the group structure of  $U(p^k)$  for prime powers  $p^k$ . When  $p$  is an

odd prime this is simple. The unit group  $U(p^k)$  has the same structure as  $C_{p^k - p^{k-1}}$  (i.e., the cyclic group of order  $p^k - p^{k-1}$ ). The case  $p = 2$  is a little more complicated. First,  $U(2)$  is the trivial group of order 1 (i.e.,  $C_1$ ),  $U(2^2)$  is cyclic order 2 (i.e.,  $C_2$ ), and finally for  $k > 2$ ,  $U(2^k)$  has the same structure as the product group  $C_2 \times C_{2^{k-2}}$ .

Considering that periods of decimal expansions are tied to the orders of elements in these group, we ask what are the possible element orders in  $U(y)$ ? A rough approximation is that elements must have orders dividing  $\varphi(y)$  (by Lagrange's Theorem). For example, since  $\varphi(60) = \varphi(2^2)\varphi(3)\varphi(5) = (4-2)(3-1)(5-1) = 16$ , we see that  $U(60)$ 's elements must have orders falling among the divisors of 16: 1, 2, 4, 8, and 16. However, carefully considering the structure of  $U(60)$  reveals that much more can be said.

From our discussion above  $U(60)$  has the same structure as  $U(4) \times U(3) \times U(5)$ , which has the structure  $C_2 \times C_2 \times C_4$ . Now cyclic groups have elements of every divisor order and orders of elements of product groups are computed using least common multiples. Thus the orders of elements of  $U(60)$  are precisely 1, 2, and 4 (so the orders 8 and 16 are not actually possible here).

Finally, what can be said in general? If  $y = 2$ , then  $U(2)$  is trivial (we only have order 1). Next, considering  $y > 2$ , factor:  $y = 2^{k_1} p_2^{k_2} \cdots p_\ell^{k_\ell}$ . Either  $y$  is divisible by an odd prime or at least  $k_1 > 1$ . Either way,  $U(y)$  must have an element of order 2. So we can ignore the  $C_2$  factor in the  $U(2^{k_1})$  piece of our decomposition of  $U(y)$ . Let  $N = \text{lcm}\{m, p_2^{k_2} - p_2^{k_2-1}, \dots, p_\ell^{k_\ell} - p_\ell^{k_\ell-1}\}$  where  $m = 2^{k-2}$  if  $k > 1$  and  $m = 1$  otherwise. Then the orders of the elements of  $U(y)$  are precisely the divisors of  $N$ .

For example, in  $U(100)$ ,  $y = 100 = 2^2 \cdot 5^2$  so  $N = \text{lcm}\{2^{2-2}, 5^2 - 5^1\} = \text{lcm}\{1, 20\} = 20$ . So elements in  $U(100)$  have orders 1, 2, 4, 5, 10, and 20.

## References

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